Apply the Newton's second law as

$$\sum \vec{F} = m\,\vec{a}_G = m\,a_C = m\ddot{x}_c$$

$$-k_1(x_c - a\theta) - k_2(x_c + b\theta) = m\ddot{x}_c$$

$$m\ddot{x}_c + (k_1 + k_2)x_c - (k_1 a - b k_2)\theta = 0 \qquad \cdots\cdots\cdots (1)$$

$$\sum \vec{M}_c = I_C \propto = I_C\ddot{\theta}$$

$$I_C\ddot{\theta} - (k_1 a - k_2 b)x_c + (k_1 a^2 - k_2 b^2)\theta = 0 \qquad \cdots\cdots\cdots (2)$$

m:- is the mass of the slab

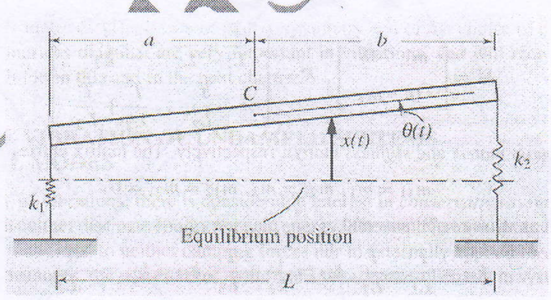
I_C:- is the mass moment of inertia of the slab about the center C.

where $\mathbf{x} = [x_c \quad \theta]^T$ is two dimensional displacement vector

Equations (1) and (2) are **equations of motion**, Equations (1) and (2) have the matrix form

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & I_c \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -(k_1 a - k_2 b) \\ -(k_1 a - k_2 b) & k_1 a^2 - k_2 b^2 \end{bmatrix}$$

Consider the simplified model of an automobile shown in Figure, let the parameter have the values $= 1500$ kg , $I_C = 2000$ kg m^2 , $k_1 = 36000$ kg/m, , $k_1 = 40000$ kg/m, $a = 1.3$ m and $b = 1.7$ m, calculate the natural modes of the system and write the expression for the response.



Solution

To calculate the natural modes, we must solve the eigenvalue problem for the system, which is based on the free vibration equations, so let $F = 0$ in Equation (7), we can write the free vibration equations in the matrix form

$$\begin{bmatrix} 1500 & 0 \\ 0 & 2000 \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 76000 & 21200 \\ 21200 & 176440 \end{bmatrix} \begin{bmatrix} x_c \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \cdots\cdots\cdots\cdots (a)$$

But, free vibration is harmonic, so that by analogy with Equations () and () we can write

$$x_c(t) = A_1 e^{i\omega t}, \qquad \theta(t) = A_2 e^{i\omega t} \qquad \cdots\cdots\cdots\cdots (b)$$

$$\ddot{x}_c(t) = -A_1\omega^2 e^{i\omega t}, \qquad \ddot{\theta}(t) = -A_2\omega^2 e^{i\omega t}$$

Then from the equations (1) and (2)

Substituting Equations (b) in Equation (a) and dividing through by $e^{i\omega t}$, we obtain the eigenvalue problem

$$-\omega^2\begin{bmatrix}1500 & 0\\ 0 & 2000\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix}+\begin{bmatrix}76000 & 21200\\ 21200 & 176440\end{bmatrix}\begin{bmatrix}A_1\\ A_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \dots\dots\dots\dots (c)$$

Hence, according to Equation (33), the characteristic Equation for the problem at hand is

$$\Delta(\omega^2)=\det\begin{bmatrix}76000-1500\omega^2 & 21200\\ 21200 & 176440-2000\omega^2\end{bmatrix}=0$$

$$=3\times10^6(\omega^4-138.887\omega^2+4.320) \quad \dots\dots\dots\dots (d)$$

which has the solution

$$\begin{matrix}\omega_1^2\\ \omega_2^2\end{matrix}=69.443\mp\sqrt{(69.443)^2-4320}$$

$$=69.443\mp22.414=\begin{cases}47.0296 & (\text{rad/sec})^2\\ 91.8571 & (\text{rad/sec})^2\end{cases} \quad \dots\dots\dots\dots (e)$$

so that the natural frequencies are

$$\omega_1=6.858 \ \text{rad/sec} \quad , \quad \omega_2=9.58421 \ \text{rad/sec} \quad \dots\dots\dots\dots (f)$$

The natural modes can be obtained by replacing ω^2 by ω_1^2 and ω_2^2 in Equation (c), To this end, we substituting $\omega^2=\omega_1^2$ in the top row of Equation (c) and write

$$(76000-1500\times47.0296)A_1+21200A_2=0 \quad \dots\dots\dots\dots (g)$$

which yields

$$\left(\frac{A_1}{A_2}\right)^{(1)}=-\frac{5455.636}{21200}=-0.257341 \quad \text{rad/m} \quad \dots\dots\dots\dots (h)$$

Similarly, introducing $\omega^2=\omega_2^2$ in the top row of Equation (c), we have

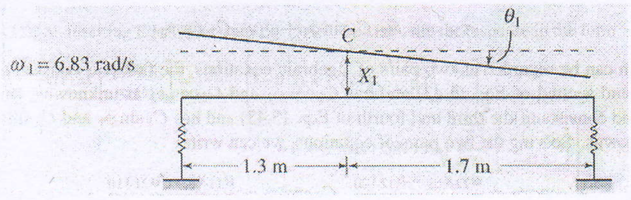
$$(76000-1500\times91.857)A_1+21200A_2=0 \quad \dots\dots\dots\dots (i)$$

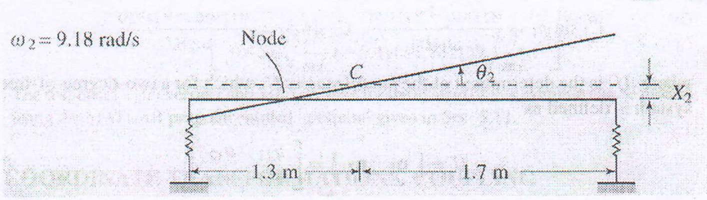
From which we obtain

$$\left(\frac{A_1}{A_2}\right)^{(2)}=\frac{61785.638}{21200}=2.9144 \quad \text{rad/m} \quad \dots\dots\dots\dots (j)$$

Hence, letting arbitrarily $A_1=1$, the natural modes become

$$\mathbf{u}_1=\begin{bmatrix}A_1\\ A_2\end{bmatrix}=\begin{bmatrix}1\\ -0.257341\end{bmatrix} \quad , \quad \mathbf{u}_2=\begin{bmatrix}A_1\\ A_2\end{bmatrix}=\begin{bmatrix}1\\ 2.9144\end{bmatrix} \quad \dots\dots\dots\dots (k)$$





H.W If the two equations of motions for the system shown in Figure (2) are

$$m\ddot{x}_1 + \frac{3mg}{l}x_1 - \frac{mg}{l}x_2 = 0$$

$$m\ddot{x}_2 - \frac{mg}{l}x_1 + \frac{mg}{l}x_2 = 0$$

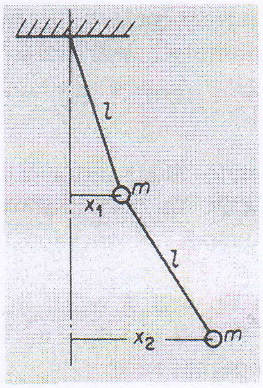


Figure (2)

Then, the natural frequencies of the system are

a- $\sqrt{0.586\,\frac{g}{l}}$ $\sqrt{4.414\,\frac{g}{l}}$

b- $\sqrt{0.686\,\frac{g}{l}}$ $\sqrt{4.414\,\frac{g}{l}}$

c- $\sqrt{0.586\,\frac{g}{l}}$ $\sqrt{3.414\,\frac{g}{l}}$

d $\sqrt{0.686\,\frac{g}{l}}$ $\sqrt{3.414\,\frac{g}{l}}$

Matrices

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where

$$i = \text{row}, \qquad j = \text{column}$$

Row matrix When the matrix has one row the matrix called row matrix

$$[B_i] = [b_1 \quad b_2 \quad b_3]$$

Column matrix or vector A column matrix has $j = 1$ called column matrix

$$[C_j] = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Unit matrix [I]

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal matrix

$$\text{diagonal matrix} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Transpose The transpose $[A_{ij}]^T$ of a matrix $[A_{ij}]$ is a matrix in which the rows and columns are interchanged i.e.,

$$[A_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \qquad [A_{ij}]^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Minor A minor M_{ij} of a matrix $[A_{ij}]$ is formed by deleting the i-th row and j-th column fro the determinant of the original matrix. If

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad then \qquad \mathrm{M}_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

Cofactor The cofactor C_{ij} is equal to the signed $(-1)^{i+j}\mathrm{M}_{ij}$, for example

$$c_{12} = (-1)^{1+2}\mathrm{M}_{12} = -\mathrm{M}_{12}$$

Adjoint matrix An adjoint matrix of a square matrix $[A_{ij}]$ is a transpose of the matrix of cofactor of $[A_{ij}]$ i.e.,

$$\text{adj}[A_{ij}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Inverse matrix

$$[A_{ij}]^{-1} = \frac{\text{adj}[A_{ij}]}{|A_{ij}|}$$

Note that $[A_{ij}]^{-1}[A_{ij}] = [\text{I}]$

Forced Harmonic Vibration of Two Degree Freedom Systems